International Grammar School

Year 12
Mathematics Extension 1
HSC Trial Examination
2015

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- In Questions 11 - 14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II

60 marks

- Attempt Questions 11-14
- Start each question in a new writing booklet
- Write your name on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your name and "N/A" on the front cover
- Allow about 1 hour and 45 minutes for this section

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is the correct domain and range for $y = 2\cos^{-1}(x-1)$?

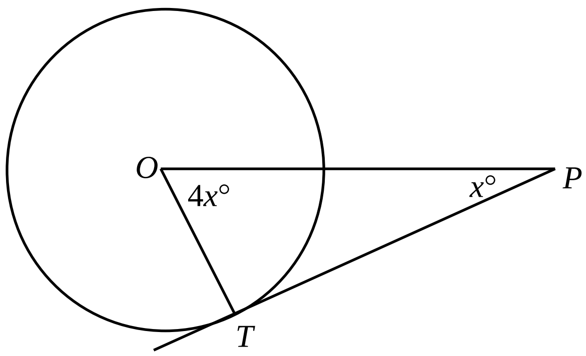
(A) Domain $0 \le x \le 2$, Range $0 \le y \le \pi$

(B) Domain $-1 \le x \le 1$, Range $0 \le y \le \pi$

(C) Domain $0 \le x \le 2$, Range $0 \le y \le 2\pi$

(D) Domain $-1 \le x \le 1$, Range $0 \le y \le 2\pi$

2 The diagram shows a circle with centre O. The line PT is a tangent to the circle at the point T. $\angle TOP = 4x°$ and $\angle TPO = x°$.



What is the value of x?

(A) 9

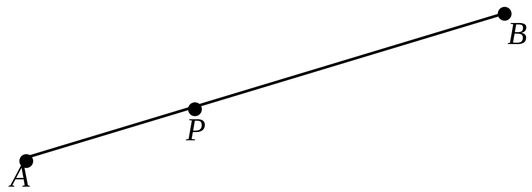
(B) 18

(C) 36

(D) 72

3 The point P divides the interval AB in the ratio 3:7.

In what external ratio does A divide the interval PB?



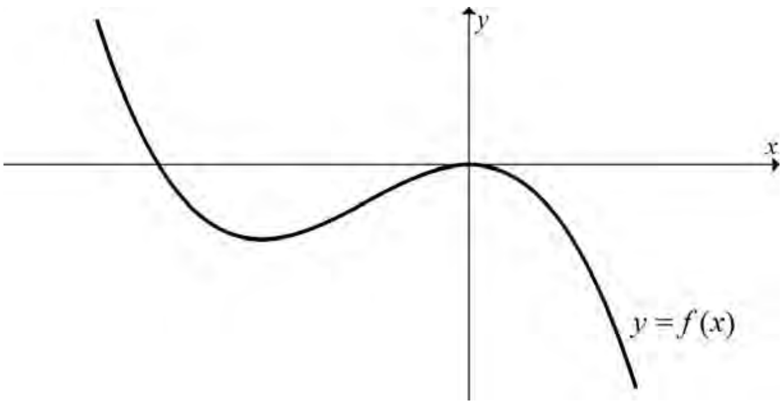
(A) 3:10

(B) 3:4

(C) 7:3

(D) 10:3

4 The diagram shows the graph of a cubic function $y = f(x)$.



What is a possible equation of this function?

(A) $f(x) = -x(x-2)(x+2)$

(B) $f(x) = x^2(x-2)$

(C) $f(x) = -x^2(x-2)$

(D) $f(x) = -x^2(x+2)$

5 Given that $t = \tan\dfrac{\theta}{2}$, which expression is equal to $\tan\theta - \tan\dfrac{\theta}{2}$?

(A) t

(B) $\dfrac{t\left(1+t^2\right)}{1-t^2}$

(C) $\dfrac{t}{1-t^2}$

(D) $\dfrac{t\left(3-t^2\right)}{1-t^2}$

6 What is the equation of the horizontal asymptote of the function $y = \dfrac{2x}{4-x}$?

(A) $x = 4$

(B) $y = 2$

(C) $x = -2$

(D) $y = -2$

7 The parametric equations of a curve are $x = p+1$ and $y = p^2 - 1$. Which of the following is the Cartesian equation of the curve?

(A) $x^2 = 4y$

(B) $y = x^2 - 2x$

(C) $y = x^2 - 2$

(D) $y = \left(x-2\right)^2$

8 Four female and four male students are to be seated around a circular table. In how many ways can this be done if the males and females must alternate?

(A) $4! \times 4!$

(B) $3! \times 4!$

(C) $3! \times 3!$

(D) $2 \times 3! \times 3!$

9 The polynomial $P(x) = x^3 + 2x + k$ has $(x-2)$ as a factor.

What is the value of k?

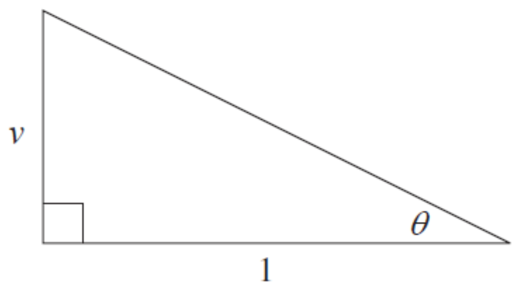
(A) -12

(B) -10

(C) 10

(D) 12

10 Consider the triangle shown below.



Given $\tan\theta = v$, $\sin 2\theta$ equals

(A) $\dfrac{2v}{\sqrt{1+v^2}}$

(B) $\dfrac{2}{\sqrt{1+v^2}}$

(C) $\dfrac{v^2-1}{1+v^2}$

(D) $\dfrac{2v}{1+v^2}$

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) Find $\lim_{x \to 0} \frac{\sin 2x}{5x}$ 1

(b) Consider the polynomial $P(x) = 2x^3 - 7x^2 + 7x + k$, where k is a constant.
The three zeroes of $P(x)$ are 1, 2 and α.
Find the value of k. 2

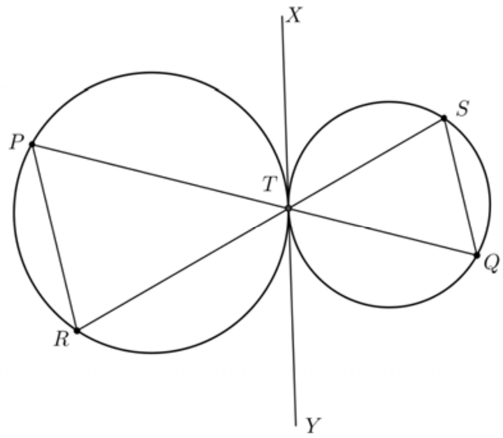
(c) Sketch the graph of $y = 2\cos^{-1}\frac{x}{6}$. 2

(d) Solve $\frac{x^2}{2-x} > 1$. 3

(e) Evaluate $\int_0^2 \frac{-1}{\sqrt{16-x^2}}\,dx$. 2

Question 11 continues over the page

(f) The diagram below shows two circles that touch at the point T. Points P, Q, R and S lie on the circles as shown.
PTQ and RTS are straight lines. XY is the common tangent at T.



Prove that PR is parallel to SQ. 2

(g) A soft drink taken from a cool room has a temperature of 3°C. It is placed in a room of constant temperature 25°C.

The temperature T of the soft drink after t minutes is given by $T = 25 - Ae^{-0.04t}$, where A is a positive constant.

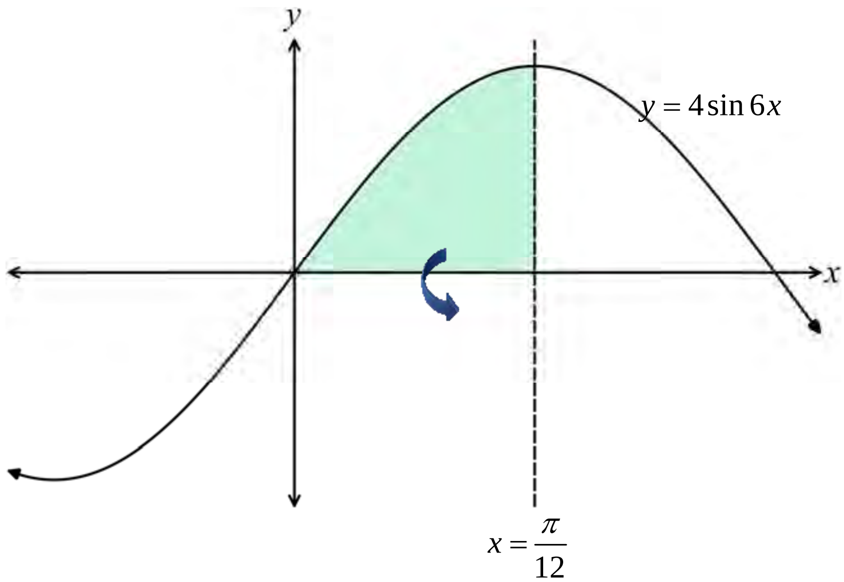
(i) How long will it take for the soft drink to reach a temperature of 15°C to the nearest minutes? 2

(ii) Find the rate at which the soft drink is increasing in temperature at 15°C. 1

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The region bounded by $y = 4\sin 6x$, the x axis and the line $x = \dfrac{\pi}{12}$ is rotated about the x axis to form a solid.



Find the exact volume of this solid. 3

(b) The acceleration of a particle moving along the x axis is $\ddot{x} = 1 + \dfrac{x}{x^2+1}$, where x is its displacement from the origin O after t seconds.

Given that the velocity $v = 2$, when $x = 2$, show that the velocity is given by $v = \sqrt{2x + \ln\left(\dfrac{x^2+1}{5}\right)}$. 3

Question 12 continues over the page

(c) A particle moves in a straight line so that its displacement x cm from the origin at time t seconds is given by $x = a\cos 4t$.

(i) Show that the particle is moving in simple harmonic motion. 1

(ii) After a time $t = 2\pi$ seconds the particle travelled a distance of 80 cm. Explain why the amplitude $a = 5$ cm. 1

(iii) What is the velocity of the particle when it passes through the origin for the first time? 2

(d) For the function $f(x) = x - \frac{1}{2}x^2$, restricting $f(x)$ to the domain $x \le 1$ allows there to be an inverse function $f^{-1}(x)$.

(i) Show that $f^{-1}(x) = 1 - \sqrt{1-2x}$ 3

(ii) Sketch $y = f^{-1}(x)$ 2

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Use mathematical induction to prove that $3^n - 2n - 1$ is divisible by 4 for all integers $n \geq 1$. 3

(b) The ODD function $f(x) = \dfrac{x}{x^2 - 1}$ has derivatives:

$$f'(x) = -\frac{(x^2+1)}{(x^2-1)^2} \text{ and } f''(x) = \frac{2x^3+6x}{(x^2-1)^3}$$ (Do NOT prove this)

(i) State the domain of $y = f(x)$. 1

(ii) Show that $y = f(x)$ has no stationary points. 1

(iii) Explain why $y = f(x)$ is always decreasing. 1

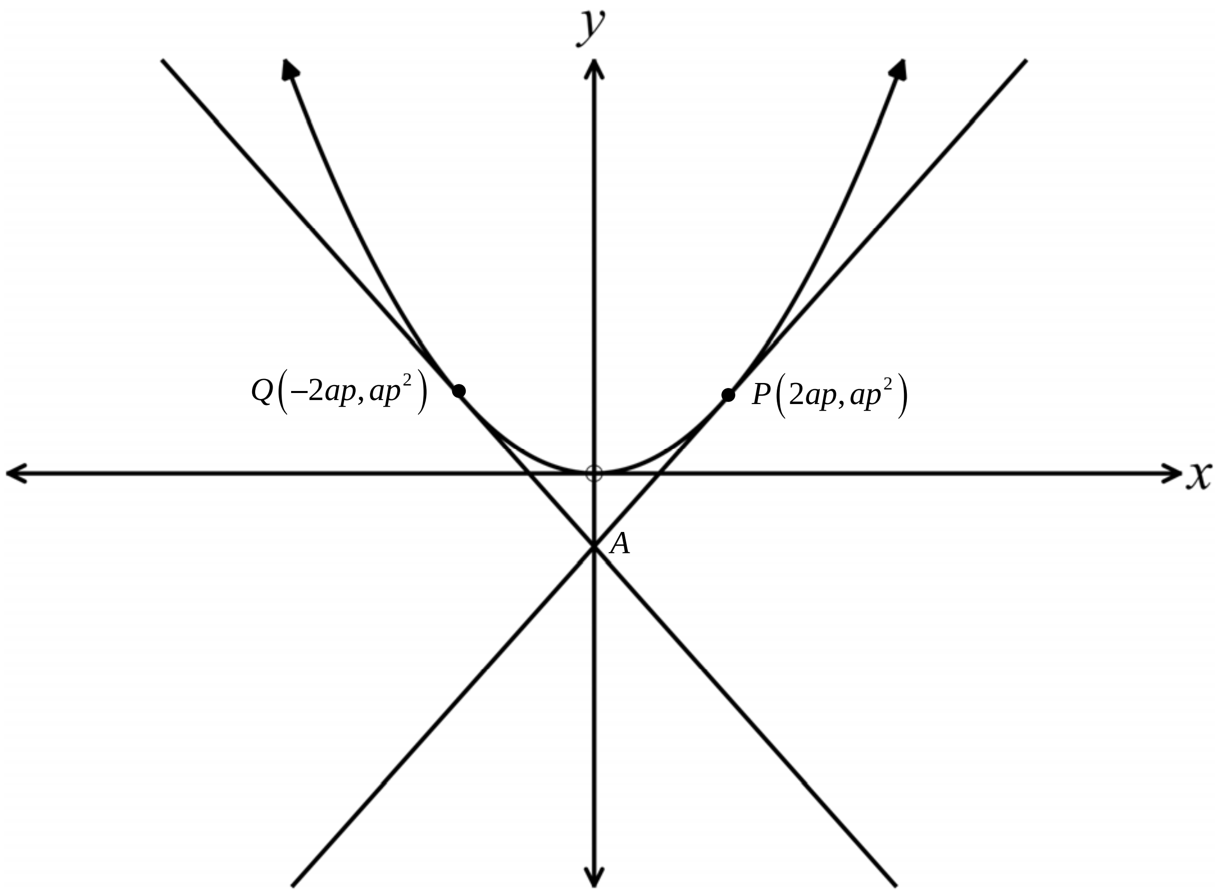
(iv) Find any point(s) of inflexion on $y = f(x)$. 1

(v) Find the value of y as $x \to \infty$. 1

(vi) Sketch $y = f(x)$ showing all important features. 2

Question 13 continues over the page

(c) The point $P\left(2ap, ap^2\right)$ and $Q\left(-2ap, ap^2\right)$ lie on the parabola $x^2 = 4ay$.
The tangents at P and Q intersect at A.



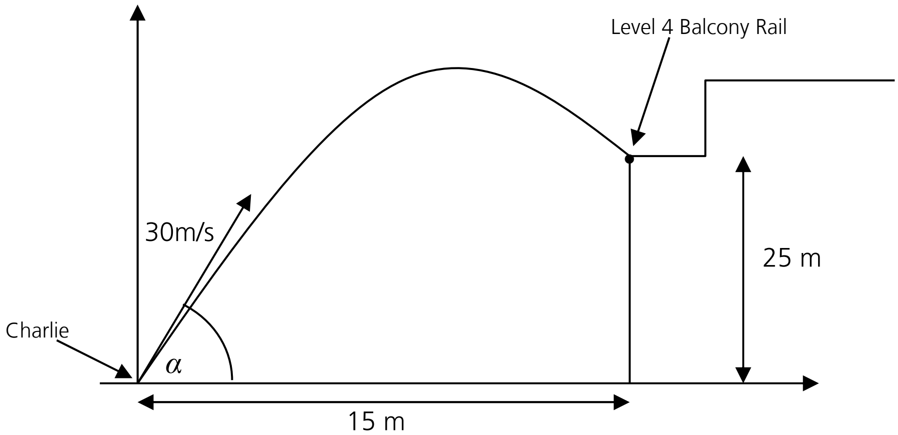
(i) Show that the coordinates of the point A are $\left(0, -ap^2\right)$. 3

(ii) M is the midpoint of PQ. State the coordinates of M. 1

(iii) Show that the origin O is the midpoint of AM. 1

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Charlie is attempting to throw a ball from the IGS canteen area to the balcony on level 4 of the Kelly Street building. Each time he throws the ball at a speed of 30 m/s and an angle of α to the ground.
The balcony on level 4 is 25 metres high and Charlie is 15 metres from the base of the building. Assume $g = 10\,\text{ms}^{-2}$.



Charlie calculates the equations of motion to be:

$$x = 30t\cos\alpha \qquad y = -5t^2 + 30t\sin\alpha \quad \text{(Do NOT prove this)}$$

(i) Show that the equation of the path of the ball is: 2

$$y = -\frac{x^2}{180}\tan^2\alpha + x\tan\alpha - \frac{x^2}{180}$$

(ii) Charlie wants to hit the rail of the level 4 balcony perfectly. Calculate the angle(s) that Charlie needs to throw the ball to accomplish this. 3

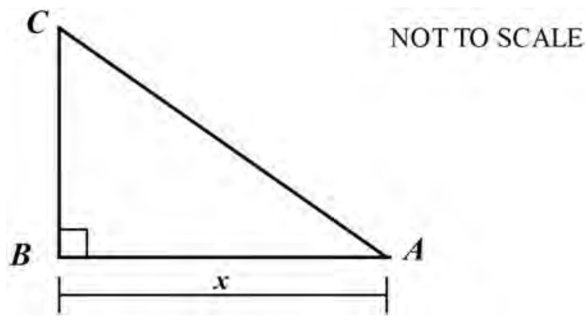
(b) (i) Show that $\sin 3x = 3\sin x - 4\sin^3 x$. 2

(ii) Hence, evaluate $\int 2\sin x - 4\sin^3 x\, dx$. 1

(c) Use an initial estimate of $x = 1$ and one application of Newton's method to solve the equation $2\sin\left(\frac{x}{2}\right) + x - \pi = 0$ to 2 decimal places. 2

Question 14 continues over the page

(d) In the triangle ΔABC below. $AB = x$ cm and $\angle ABC = 90°$.



(i) Show that the perimeter of ΔABC is given by the equation:

$$P = x(1 + \sec A + \tan A)$$ 2

(ii) If $x = 20$ cm and $\angle A$ is increasing at a constant rate of 0.1 radians/second, find the rate at which the perimeter of the triangle is increasing when $\angle A = \frac{\pi}{6}$ radians. 3

End of Examination

Year 12 Mathematics E1 Trial 2015

1) D: $-1 \le x-1 \le 1$

$0 \le x \le 2.$

R: $0 \le y \le 2\pi$

$\boxed{C}$

2) $\angle PTO = 90$

$5x = 90$

$x = 18^\circ$

$\boxed{B}$

3)

3:10

$\boxed{A}$

4) $\boxed{D}$

5) $\dfrac{2t}{1-t^2} - t = \dfrac{2t - t(1-t^2)}{1-t^2}$

$= \dfrac{t^3+2t-t}{1-t^2}$

$= \dfrac{(t^3+t)}{1-t^2}$ $\boxed{B}$

$= \dfrac{t(t^2+1)}{1-t^2}$

(2)

6) $\lim_{x \to \infty} \dfrac{\frac{2x}{x}}{\frac{4}{x} - \frac{x}{x}} = \dfrac{2}{-1}$

$y = -2$

$\boxed{D}$

7) $\cancel{x} p = x - 1$

$y = (x-1)^2 - 1$

$= x^2 - 2x + 1 - 1$

$= x^2 - 2x.$

$\boxed{B}$

8) $\boxed{B}$

9) $\alpha, \beta, 2$

$\alpha + \beta + 2 = 0 \qquad \alpha + \beta = -2$

$\alpha\beta + 2\alpha + 2\beta = +2$

$\alpha\beta - 4 = 2$

$\alpha\beta = 6$

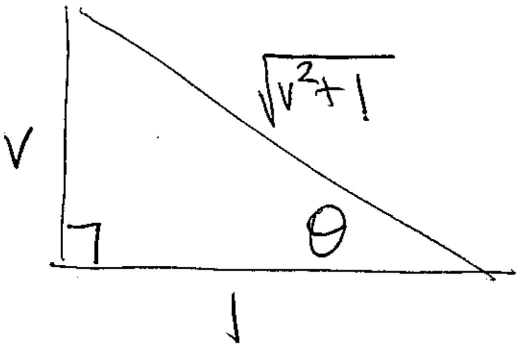
$\therefore \alpha\beta\gamma = 6 \times 2$

$= 12$

$\therefore k = -12$

$\boxed{A}$

10) 3



$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2 \times \frac{v}{\sqrt{v^2+1}} \times \frac{1}{\sqrt{v^2+1}}$$
$$= \frac{2v}{v^2+1}$$

$\boxed{D.}$

Question 11

4

a) $\frac{2}{5}$

✓ correct

b) $\alpha + 3 = \frac{7}{2}$

$\alpha = \frac{1}{2}$

$\therefore 1 \times 2 \times \frac{1}{2} = \frac{-k}{2}$

$k = -2$

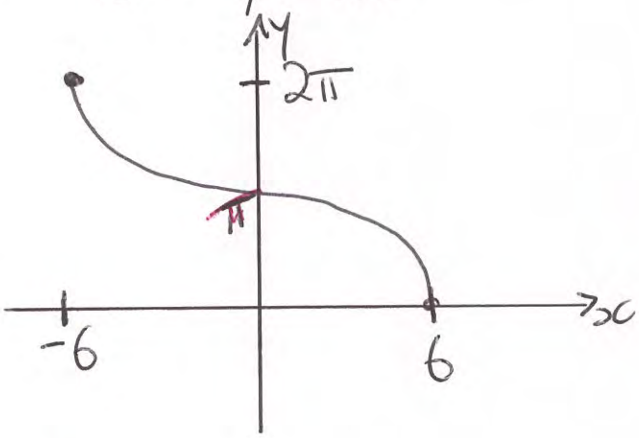
✓ correct α

(✓ for correct k any other method)

✓

c) D: $-6 \le x \le 6$

R: $0 \le y \le 2\pi$



✓ shape/domain

✓ domain/range y-intercepts correct.

d) $\frac{x^2}{2-x} > 1 \quad x \neq 2$

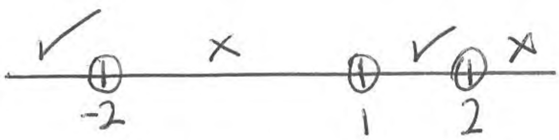
$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$

✓ to quadratic equation



$x < -2,\ 1 < x < 2$

✓ each solution

e) $\int_0^2 \frac{-1}{\sqrt{16-x^2}}\,dx = \left[\cos^{-1}\frac{x}{4}\right]_0^2$

✓ correct integration

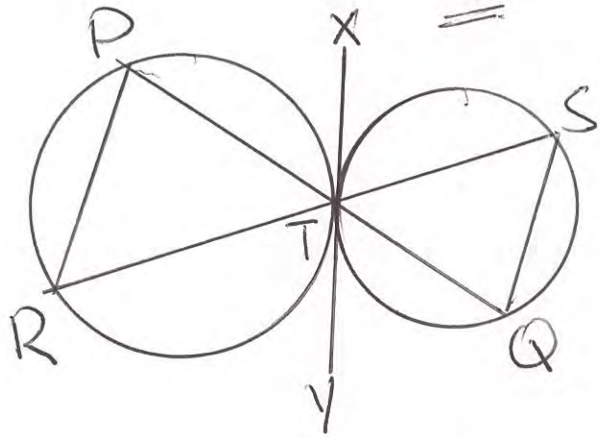
$= \cos^{-1}\frac{1}{2} - \cos^{-1}0$

$= \frac{\pi}{3} - \frac{\pi}{2}$

$= -\frac{\pi}{6}$

✓ correct answer.

f)



$\angle SQT = \angle STX$ ($\angle$ in alt segment)
$\angle STX = \angle YTR$ (vert opp)
$\angle YTR = \angle TPR$ ($\angle$ in alt segment)

✓ correct with correct circle geom reason.

$\therefore \angle TPR = \angle SQT$

$PR \parallel SQ$ (alt $\angle$'s on $\parallel$ lines)

✓ reason

g) $A = 22$

$\therefore T = 25 - 22e^{-0.04t} = 15$

$\frac{10}{22} = e^{-0.04t}$

✓ correct expression for e or correct ln.

(5)

6

$$\ln\frac{5}{11} = -0.04t$$

$$t = \frac{\ln\frac{5}{11}}{-0.04}$$

$$= 19.7\text{ min}$$

$$= \underline{20\text{min}}$$

✓ correct solution

ii) $$\frac{dT}{dt} = 0.04 \times 22e^{-0.04t}$$

$$= 0.04(25-15)$$

$$= \underline{0.4°C/\text{min}}$$

✓ correct answer.

Question 12

(7)

a) $V = \pi \int_0^{\frac{\pi}{12}} 16\sin^2 6x\,dx$

$= 16\pi \int_0^{\frac{\pi}{12}} \sin^2 6x\,dx$ ✓

$= 16\pi \int_0^{\frac{\pi}{12}} \frac{1}{2}(1-\cos 12x)\,dx.$ ✓

$= 8\pi \left[x - \frac{\sin 12x}{12}\right]_0^{\frac{\pi}{12}}$

$= 8\pi \left[\frac{\pi}{12} - 0 - 0\right]$

$= \frac{2\pi^2}{3}$ ✓

1. correct volume integral
2. correct rearrangement involving $\cos 12x$.
3. Correct answer.

b) $\frac{1}{2}v^2 = \int 1 + \frac{x}{x^2+1}\,dx$

$= x + \frac{1}{2}\ln(x^2+1) + C$ ✓

$v^2 = 2x + \ln(x^2+1) + C$

@ $v=2,\ x=2$

$4 = 4 + \ln 5 + C$

$C = -\ln 5$ ✓

$\therefore v^2 = 2x + \ln(x^2+1) - \ln 5$

$v^2 = 2x + \ln\left(\frac{x^2+1}{5}\right)$

or v -

Since $v > 0$ for $x > 0$

$v = \sqrt{2x + \ln\left(\frac{x^2+1}{5}\right)}$ ✓

Q6.
1. Correct integration $\frac{1}{2}v^2$ or v^2.
2. Correctly solving for C.
3. Correct expression with explanation for $\pm$

8

c i) $x = a\cos 4t$

$\dot{x} = -4a\sin 4t$

$\ddot{x} = -16a\cos 4t$

$= -16x$

$\therefore$ in SHM $\ddot{x} \propto x$. ✓

1, Correct $\ddot{x}$ with x showing $\ddot{x} \propto -x$. or other equivalent explanation.

ii) $T = \frac{2\pi}{4} = \frac{\pi}{2}$

Particle went through 4 complete wavelengths

$\therefore$ Amplitude $= 80 \div 4 \div 4$

$= 5\text{cm}$. ✓

1, Correct explanation that uses period.

iii) @ $x=0$, $t=?$

$5\cos 4t = 0$

$4t = \frac{\pi}{2}$

$t = \frac{\pi}{8}$ ✓

$\dot{x} = -20\sin 4t$

$= -20\sin\frac{\pi}{2}$

$= -20$ m/s ✓

$\therefore$ 20 m/s in the negative direction

1, Correctly finding time for $x=0$ $(t=\frac{\pi}{8})$

2, Correct answer

d i)

$$y = x - \tfrac{1}{2}x^2 \qquad x(1 - \tfrac{1}{2}x)$$

$$x = y - \tfrac{1}{2}y^2$$

$$2x = 2y - y^2$$

$$y^2 - 2y = -2x$$

$$y^2 - 2y + 1 = -2x + 1$$

$$(y-1)^2 = 1 - 2x$$

$$y - 1 = \pm\sqrt{1-2x}$$

Since $x \leq 1$ for $f(x)$, $y \leq 1$ for $f^{-1}(x)$

$$\therefore y - 1 = -\sqrt{1-2x}$$

$$y = 1 - \sqrt{1-2x}$$

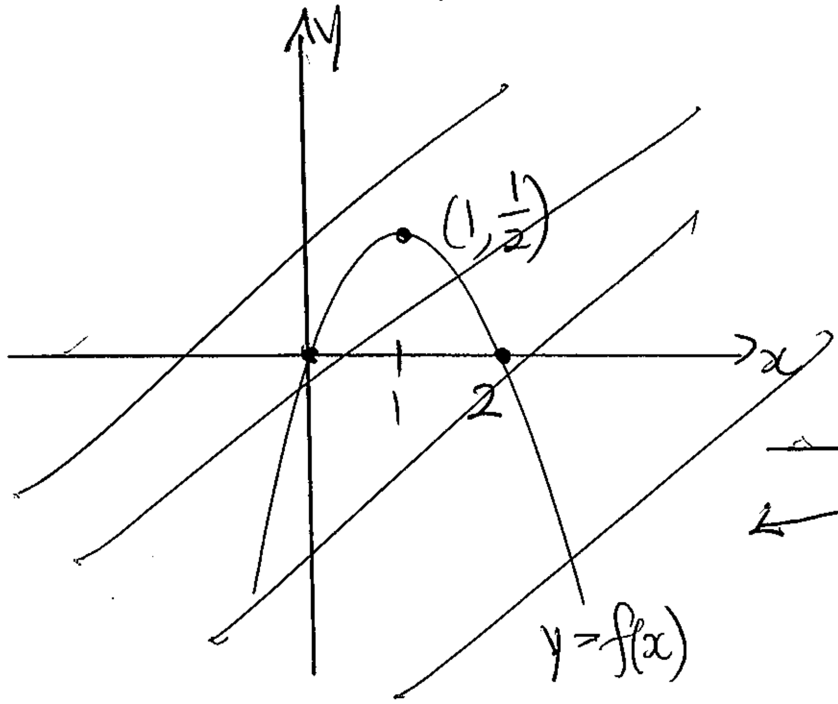
9

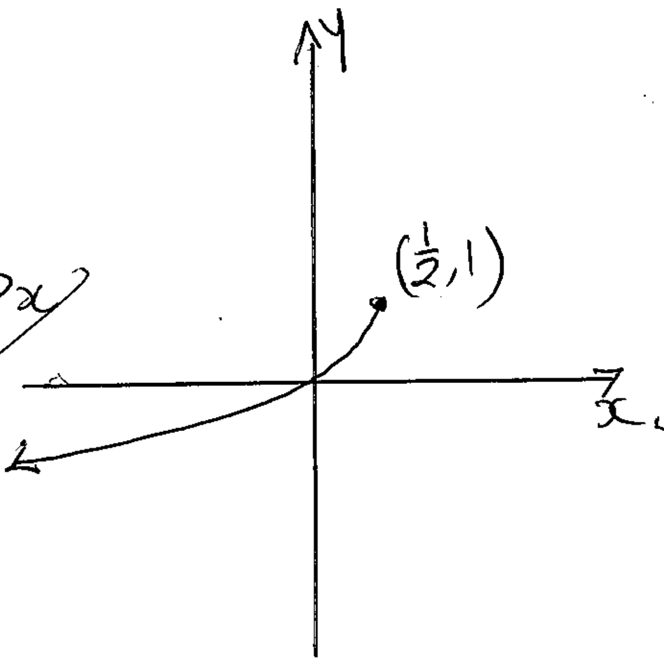
1, Correct swap of x/y

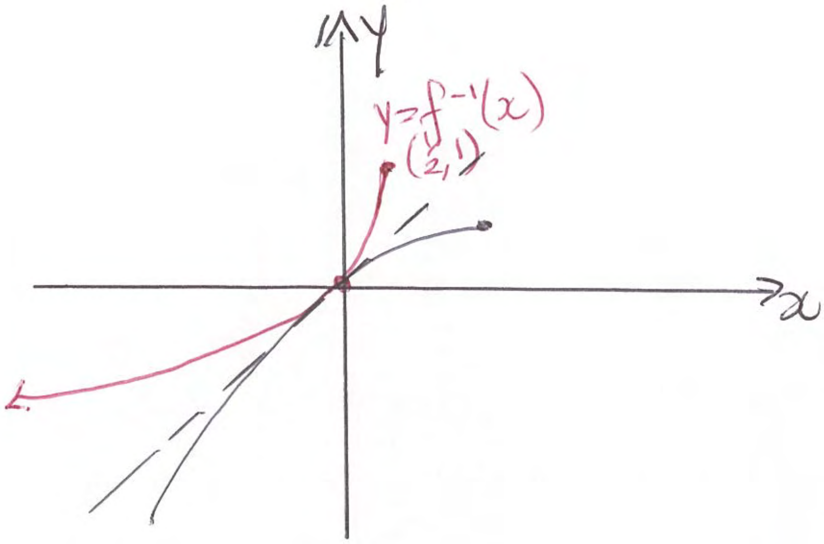
2, Correct use of completing the square.

3, Correct rearrangement that explains $\pm$ and which to use.

ii)







10

1. Correct shape. ✓

2. Correct end point $(\frac{1}{2}, 1)$ and intercept $(0,0)$. ✓

a) $3^n - 2n - 1 = 4P$.

i) $n = 2$.

$9 - 4 - 1 = 4$ $\therefore$ true $n = 2$. ✓

1 mark $n = 2$ correct

ii) $n = k$

$3^k - 2k - 1 = 4Q$

iii) $n = k + 1$

$3^{k+1} - 2(k+1) - 1 = 3.3^k - 2k - 3$ ✓

1 mark correct use assumption algebraically.

$= 3(3^k - 2k - 1) + 6k + 3 - 2k - 3$ by assumption

$= 3 \times 4Q + 4k$

$= 4(12Q + k)$ $\therefore$ divisible by 4 ✓

1 mark showing result algebraically correct.

$\therefore$ true by mathematical Induction.

1b)

iii) $f(x) = \dfrac{x}{x^2+1}$

$= \dfrac{x}{(x+1)(x-1)}$ $\quad x \neq \pm 1$

i) $\forall x \in \mathbb{R},\ x \neq \pm 1$. ✓

1 mark both ± 1

ii) $f'(x) = -\dfrac{(x^2+1)^2}{(x^2-1)^2}$

SP $f'(x) = 0$ but $-(x^2+1)^2 \neq 0$ ✓

$\therefore$ no SP

1 mark reasonable + correct explanation

iii) ~~f~~ decreasing $f'(x) < 0$

$\dfrac{(x^2+1)^2}{(x^2-1)^2} > 0\ \forall x$ so $-\dfrac{(x^2+1)^2}{(x^2-1)^2} < 0$ ✓

1 mark reasonable + correct explanation

$\therefore$ always decreasing

iv) $f''(x) = \dfrac{2x^3 + 6x}{(x^2-1)^3}$

$f''(x) = 0$ $\quad 2x(x^2 + 3) = 0$

$x^2 + 3 \neq 0$ so $x = 0$.

1 mark correct with concavity test ✓

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
$f''(x)$	>0	0	<0

$\therefore$ concavity changes.

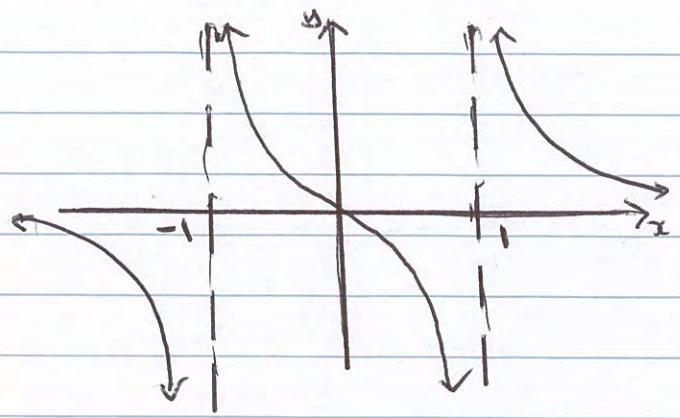
$x = 0$ $f(0) = 0$ $\therefore$ POI $(0, 0)$

v) $x \to \infty$, $f(x) \to 0$.

$\therefore y = 0$

1 mark correct

vii).



1 mark VAs

1 mark POI correct + correct shape.

c) i) $y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$

at $P(2ap, ap^2)$ $m_T = p$

eq tang: $y - ap^2 = p(x - 2ap)$

$= px - 2ap^2$

$px - y = ap^2$ ✓

eq of tangent at P correct

similarly at $Q(-2ap, ap^2)$

$m_T = -p$.

$y - ap^2 = -p(x + 2ap)$

$y - ap^2 = -px - 2ap^2$

$px + y = -ap^2$ ✓

eq of tangent at Q correct.

for (A). $px - y = ap^2$ ①

$px + y = -ap^2$ ②

① + ② $2px = 0$

$x = 0$ ✓

solving for point of intersection

Note: A described in question as point of intersection

② − ① $2y = -2ap^2$

$y = -ap^2$ $\therefore A(0, -ap^2)$

ii) M : $(0, ap^2)$ ✓

correct midpt.

iii) Midpt AM $\left(\frac{0+0}{2}, \frac{-ap^2+ap^2}{2}\right) = (0,0)$ ✓

Question 14

(14)

ai) $t = \dfrac{x}{30\cos\alpha}$.

$\therefore y = -5\left(\dfrac{x}{30\cos\alpha}\right)^2 + 30\sin\alpha\left(\dfrac{x}{30\cos\alpha}\right)$ ✓ correct sub into y.

$= \dfrac{-5x^2}{900\cos^2\alpha} + x\tan\alpha$.

$= \dfrac{-5x^2}{900}(1+\tan^2\alpha) + x\tan\alpha$.

reaching result correctly

$= \dfrac{-x^2}{180}\tan^2\alpha + x\tan\alpha - \dfrac{x^2}{180}$ ✓

ii) $x = 15,\ y = 25$

$25 = \dfrac{-(15)^2}{180}\tan^2\alpha + 15\tan\alpha - \dfrac{15^2}{180}$ ✓ correct sub for x.

$25 = -\dfrac{5}{4}\tan^2\alpha + 15\tan\alpha - \dfrac{5}{4}$

$20 = -\tan^2\alpha + 12\tan\alpha - 1$

$\tan^2\alpha - 12\tan\alpha + 21 = 0$ ✓ correct quadratic eq.

$$\tan\alpha = \frac{12 \pm \sqrt{144 - 4 \times 21}}{2}$$

$$= \frac{12 \pm \sqrt{60}}{2}$$

$$= 6 \pm \sqrt{15}$$

$$\therefore \alpha = \underline{84^\circ 13', 64^\circ 49'}$$

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both correct

bi) $\sin 3x = \sin(2x + x)$

$$= 2\sin x \cos x . \cos x + \cos 2x . \sin x$$

$$= 2\sin x \cos^2 x + \sin x(1 - 2\sin^2 x)$$

$$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$

$$= \underline{3\sin x - 4\sin^3 x}$$

correct use sum of angles

reaching result with correct working.

ii) $\int 2\sin x - 4\sin^3 x \, dx = \int 3\sin x - 4\sin^3 x - \sin x \, dx$

$$= \int \sin 3x - \sin x \, dx$$

$$= \underline{-\frac{\cos 3x}{3} + \cos x + C}$$

Correctly using result to reach answer

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c) $f(x) = 2\sin\frac{x}{2} + x - \pi$

$f'(x) = \cos\frac{x}{2} + 1$

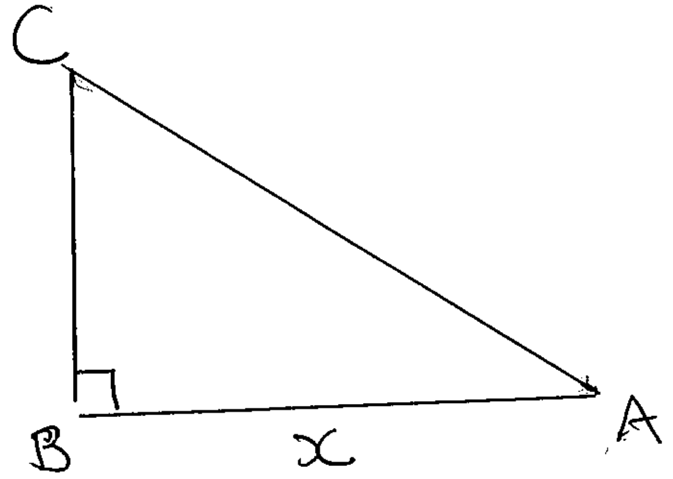
$\therefore x = 1 - \dfrac{2\sin\frac{1}{2} + 1 - \pi}{\cos\frac{1}{2} + 1}$

$= \underline{\underline{1.63}}$

✓ correct use of formula

✓ correct

d)



$BC = x\tan A$

$AC = \dfrac{x}{\cos A} = x\sec A$

✓ finding correct expression for BC + AC

$\therefore P = x + x\tan A + x\sec A$

$= \underline{\underline{x(1 + \tan A + \sec A)}}$

reaching result correctly ✓

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ii) $\frac{dP}{dt} = \frac{dP}{dA} \times \frac{dA}{dt}$

$\frac{dP}{dA} = x\left(-(\cos A)^{-2} \cdot -\sin A + \sec^2 A\right)$

$= x \cdot \left(\frac{\sin A}{\cos^2 A} + \frac{1}{\cos^2 A}\right)$

$= x\left(\frac{\sin A + 1}{\cos^2 A}\right)$

correct derivative.

$\frac{dP}{dt} = x\left(\frac{\sin A + 1}{\cos^2 A}\right) \times 0.1$

$= 20 \times \left(\frac{\sin\frac{\pi}{6} + 1}{\cos^2\frac{\pi}{6}}\right) \times 0.1$

correct use of related rates with given information.

$= 20 \times \left(\frac{1.5}{0.75}\right) \times 0.1$

$= \underline{\underline{4 \text{ cm/s}}}$

correct answer obtained.